University of Zurich
Institute for Empirical Research in Economics

Advanced Portfolio Theory

NHH-Bergen

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3. Foundations from Asset Pricing Theory

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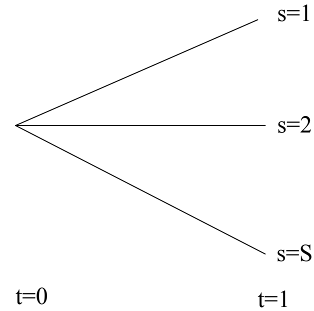
a) Asset Pricing Model

2 periods t=0,1.

S states, s=1,…,S.

Market structure A= $\begin{pmatrix} a_1^1 & \cdots & a_1^K \\ \vdots & \ddots & \vdots \\ a_S^1 & \cdots & a_S^K \end{pmatrix}$.



$$\underset{\lambda}{Max} \quad U^i(c_0^i, c_1^i, \ldots, c_S^i)$$

$$c_0^i = \lambda_0^i w^i, c_s^i = \sum_{k=1}^{K} a_s^k \frac{\lambda_k^i}{q^k} w^i, s = 1, \ldots, S.$$

$$\sum_{k=0}^{K} \lambda_k^i = 1$$

b) Axiom 0 and No-Arbitrage

Axiom 0: $c^i > d^i \Rightarrow U^i(c^i) > U^i(d^i)$

Fundamental Theorem of Asset Pricing:

If Axiom 0 implies $\exists \pi^* \in \Delta_{++}^S : R^f = \sum_{s=1}^{S} \pi_s^* R_s^k, k = 1, \ldots, K.$

Where $R_s^k = \frac{a_s^k}{q^k}$ *is the gross return* of asset k and R^f is the gross risk-free rate

Application: Pricing of Derivatives

$$A^j = \sum_{k=1}^{K} \alpha^k A^k \Rightarrow q^j = \sum_{k=1}^{K} \alpha^k q^k$$

Summers (1985): On Economics and Finance, JoF 40, page 634.

Financial Economists are like ketchup economists who are mainly concerened about the prices of different-sized bottles of ketchup. They seem to ignore what seems to many to be the more important question of what determines the overall level of asset prices.

Proof of the FTAP

Consider

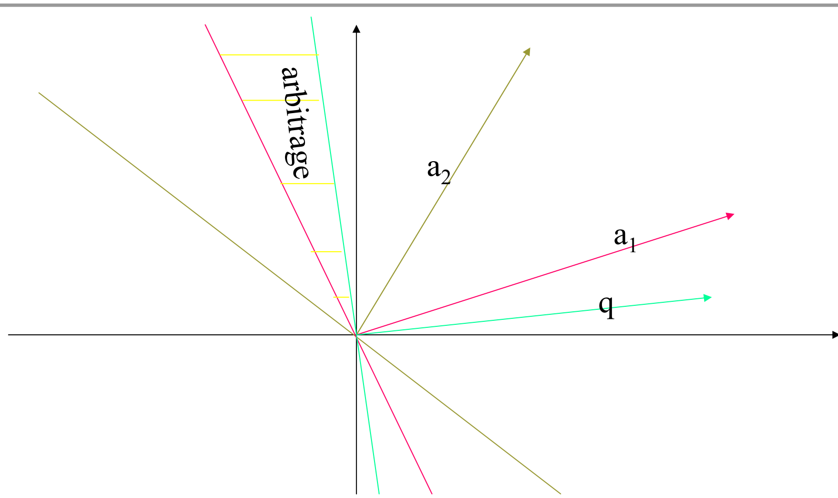
$$\max_{\theta} \quad U^i(c_0^i, c_1^i, \ldots, c_S^i)$$

$$c_0^i = w^i - \sum_k q_k \theta_k, \quad c_s^i = w_s^i + \sum_{k=1}^{K} a_{sk}\theta_k, s = 1, \ldots, S.$$

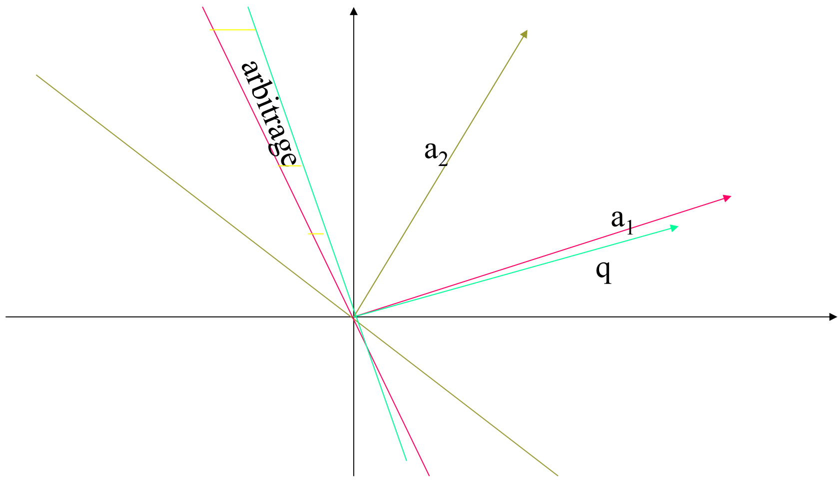
$$There \quad is \quad no \quad \theta \in R^K such \quad that \begin{bmatrix} -\vec{q}' \\ A \end{bmatrix} \theta > 0$$

$$Is \quad equivalent \quad to \quad \exists \pi \in R^S_{++} : q_k = \sum_{s=1}^{S} a_{sk}\pi_s, k = 1, \ldots, K.$$

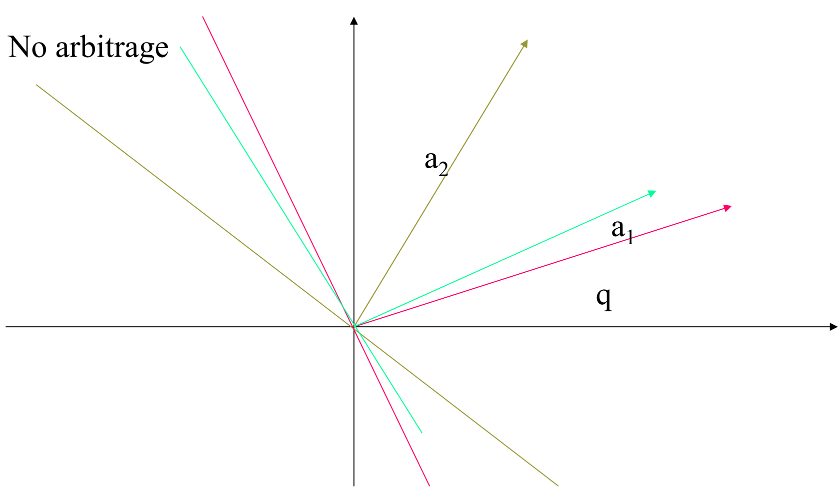
Proof of FTAP



Proof of FTAP



Proof of FTAP



FTAP

$$There\ \ is\ \ \pi \in R^S_{++} : q_k = \sum_{s=1}^{S} a_{sk}\pi_s, k = 1,\dots,K.$$

$$Applied\ \ to\ \ riskless\ \ asset:\ \ k = 1 : q_1 = \frac{1}{R^f} = \sum_{s=1}^{S} \pi_s$$

$$Hence\ \ q_k = \frac{1}{R^f}\sum_s \pi^*_s a_{s,k}\ \ where\ \ \pi^*_s = \frac{\pi_s}{\Sigma_z \pi_z}.$$

$$Thus\ \ q_k = \tfrac{1}{R^f} E_{\pi^*}(a^k)\ \ or\ \ equivalently\ \ R^f = E_{\pi^*}(\frac{a^k}{q^k}).$$

Proof of FTAP via LP-Duality:

Consider the pair of Linear Programs (LP):

$$\min_{\theta} q\theta \qquad\qquad \max_{\pi \ge 0} \pi 0$$

$$s.t.\ \ A\theta \ge 0 \qquad\qquad \mathrm{A}^{\mathrm{T}}\pi = q$$

The first LP has a solution iff there is no arbitrage.

From the LP-duality theorem then no arbitrage
is equivalent to having a solution to the second LP.

c) Risk and Return

We have seen that No-arbitrage implies: $R^f = E_{\pi^*}(\frac{a^k}{q^k}) = E_{\pi^*}(R^k).$

Change of measure:

$$\mathrm{R}^{\mathrm{f}} = E_{\pi^*}(R^k) = \sum_s \pi_s^* R_s^{\ k} = \sum_s p_s(\frac{\pi_s^{\ *}}{p_s}) R_s^{\ k} = \sum_s p_s l_s R_s^{\ k} = E_p(l\ \mathrm{R}^{\mathrm{k}})$$

By the definition of COV we get:

$$\mathrm{R}^{\mathrm{f}} = E_p(l\ \mathrm{R}^{\mathrm{k}}) = E_p(l) E_p(R^k) + COV_p(l, R^k).$$

Hence there is always a risk-return decomposition:

$$R^f = E_p(R^k) + COV_p(l, R^k)$$

d) Change of Measure

$$E_P(x) \neq E_\pi(x)$$

Distinguish between P and π !

P : physical Measure

π : equivalent Martingale-Measure

$$\pi \neq P$$

Example: Change of Measure

Example:

P physical measure

π martingale measure

$\pi = 0.2$

$P = 0.9$

$S_t = 5$

$S_{t+1}(up) = 9$

$S_{t+1}(down) = 4$

$(1-\pi) = 0.8$

$(1-P) = 0.1$

$$E_P(s_{t+1}) = 8.5$$

$$E_\pi(s_{t+1}) = 5$$

Campbell (2000), Asset Pricing at the Millenium, JoF

The challenge now is to understand the economic forces that determine the stochastic discount factor, or to put it another way, the rewards that investors demand for bearing particular risks.

e) Identifying the Likelihood Ratio Process

CAPM $\ell = \omega$ ω market portfolio

APT $\ell = \sum_{j=1}^{J} \alpha^{j} . f^{j}$

Macrofinance $\ell = U'(\omega)$

hence: $\mu_P(R^k) - R^f = \sum_{j=1}^{J} \alpha^j \beta^{kj} (\mu(f^j) - R^f)$

$$\beta^{kj} = \frac{\mathrm{cov}_P(f^j, R^k)}{VAR_P(f^j)}$$

f) Financial Econometrics

- Cross Section: APT-factors
- Time Series: Excess Return of Market

Consider SML: $\mu_k - R_f = \beta_k(\mu_M - R_f), k = 0,1,\ldots,K$

Applied for k=M gives: $\mu_M - R_f = \beta_M(\mu_M - R_f)$
This is a tautology, since $\beta_M = 1$.

SML does not tell us anything about the excess return of the market.
SML is a "cross section" result.

APT-Factors in empirical studies

Autoren	Jahr	Identifizierte Faktoren
Nai-Fu Chen Richard Roll Stephen Ross	1986	1. Wachstumsrate der industriellen Produktion 2. Inflationsrate 3. Spread zwischen kurz- und langfristigen Zinsraten 4. Default-Risikoprämie von Bonds
Michael Berry Edwin Burmeister Marjorie McElroy	1988	1. Inflationsrate 2. Spread zwischen kurz- und langfristigen Zinsraten 3. Default-Risikoprämie von Bonds 4. Wachstumsrate der aggregierten Verkäufe 5. Return auf dem S&P 500

APT-Factors in empirical studies (cont‘d)

Salomon Brothers	1990	1. Inflationsrate 2. Wachstumsrate des Bruttosozialprodukts 3. Zinsrate 4. Änderungsrate der Ölpreise 5. Wachstumsrate der Rüstungsausgaben
Mei Jianping	1993	1. Januar-Dummy Variable 2. Return auf einem value-weighted Portfolio 3. One-Month-Treasury-Bill Rate 4. Differenz zwischen One-Month-Treasury-Bill Rate und langfristiger AAA-Corporate-Bond Rate 5. Dividend-Yield auf dem value-weighted Portfolio
Eugene F. Fama Kenneth R. French	1993	1. Überrendite eines diversifizierten Marktportfolios 2. Differenz des Returns zwischen einem Portfolio aus Aktien kleiner Firmen und einem aus Aktien grosser Firmen 3. Differenz des Returns zwischen einem Portfolio aus Aktien, welche eine hohe Book-to-Market Ratio haben und einem aus Aktien, die eine tiefe Book-to-Market Ratio aufweisen

APT-Factors in empirical studies (cont'd)

E. J. Elton M. J. Gruber J. Mei	1994	1. Differenz zwischen dem Return von langfristigen Government-Bonds und kurzfristigen Treasury-Bills 2. Änderung im Return von Treasury-Bills 3. Änderung in der Wechselkursrate Dollar/Fremdwährung 4. Änderung in der Prognose des Bruttosozialprodukts 5. Änderung in der Prognose der Inflation 6. Anteil des Returns des Marktes, der nicht durch die ersten fünf Faktoren erklärt werden kann
James L. Davis	1994	1. Book-to-Market-Equity 2. Cash-Flow/Price Ratio 3. Earnings/Price Ratio
Josef Lakonishok Andrei Shleiter Robert W. Vishny	1994	1. Earnings/Price Ratio 2. Cash-Flow/Price Ratio 3. Sales-Growth Variable

APT-Factors in empirical studies (cont'd)

Gallati	1994	1. Euromarkt SFR Ein-Monats-Zinssatz 2. Obligationen-Index EFFAS Schweiz 3. Euromarkt DM Drei-Monats-Zinssatz 4. FTSE 100 Index
P. Kothari Jay Shanken Richard G. Sloan	1995	1. Beta 2. Firmengrössen-Variable

Quelle: Semesterarbeit Marc Arnold

Macrofinance

Transitory movements in aggregate (human and non-human) household wealth (a proxy for the log consumption-wealth ratio) can predict (excess) stock returns.

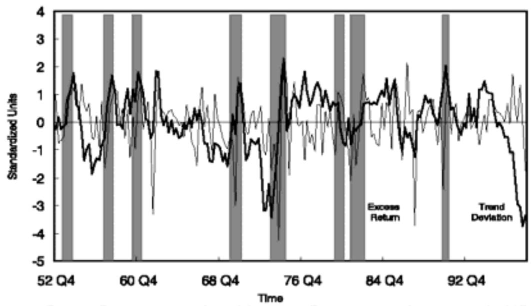


Figure 1. Excess returns and trend deviations. Excess return is the return on the S&P Composite Index less the return on the three-month Treasury bill rate. Trend deviation is the estimated deviation from the shared trend in consumption c, labor income y, and asset wealth a: $\widehat{cay}_t = c_t - \hat{\beta}_a a_t - \hat{\beta}_y y_t$. Both series are normalized to standard deviations of unity. The sample period is fourth quarter of 1952 to third quarter of 1998. Shaded areas denote NBER recessions.

Lettau Ludvigson (2001):"Consumption, Aggregate Wealth and Expected Stock Returns", The Journal of Finance, Vol. LVI No 3, pp. 815-849.

g) Equivalent Formulations of No-Arbitrage

- Prices are arbitrage free
- Martingale Property
- Discounted Dividends Model for Stocks
- Prices fluctuate trandomly
- There are no excess returns
- There are no expected gains
- There is always a risk-return decomposition

No-Arbitrage Principle (1)

(0) The price process is arbitrage-free

There exists no strategy $\theta_t, \ t = 1,2,..,T$

which generates risk-free returns on $q_t, \ t = 1,2,..,T$

(1) The process is a martingale

$$q_t^k = \frac{1}{1+r_t} E_{\pi_t}(D_{t+1}^k + q_{t+1}^k) \ \ t = 1,2,..,T$$

(2) Discounted Divident Model (DDM)

$$q_t^k = E_{\pi_t}\left(\sum_{\tau=t+1}^{T} \left(\frac{1}{1+r} \right)^{\tau-t} D_\tau^k \right), \ \ t = 1,2,..,T$$

No-Arbitrage Principle (2)

(3) Prices fluctuate randomly

$$E_{\pi_t}(q_{t+1}^k - q_t^k) = \frac{1}{1+r_t} E_{\pi_t}(D_{t+2}^k - D_t^k)$$

(4) There are no excess returns

$$E_{\pi_t}(\gamma_{t+1}^k - r_t) = 0$$

where $\gamma_t^k = \dfrac{D_{t+1}^k + q_{t+1}^k - q_t^k}{q_t^k}$

No-Arbitrage Principle (3)

(5) There are no expected gains

$$E_{\pi_t}(g_{t+1} - g_t) = 0$$

where $g_t = \sum_{\tau=t}^{T} \left[\frac{1}{1+r_t}(D_{\tau+1}^{k} + q_{\tau+1}^{k}) - q_{\tau}^{k} \right] \theta_{\tau}^{k}$

„Nobody can beat the market"

Nobody can beat the Market?

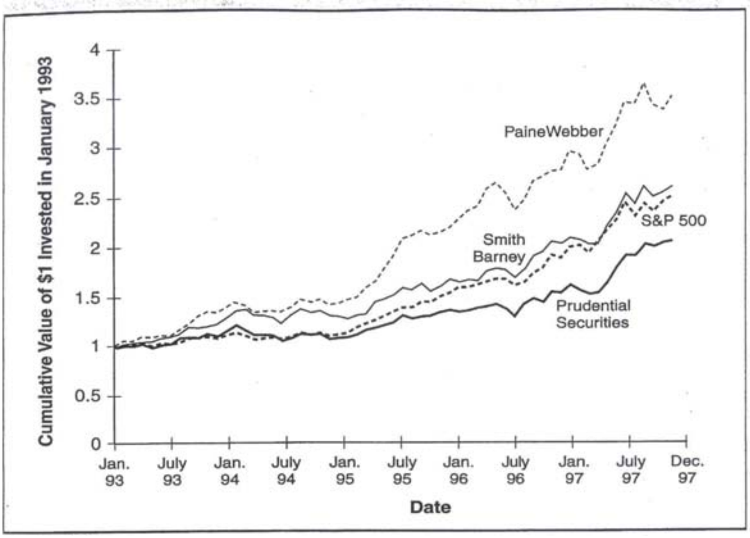
Every time somebody beats the market:

Demand and Supply:

$$\sum_{i=1}^{I} \underbrace{\sum_{k=1}^{K} q_k \theta_k^i}_{W^i} = \sum_{k=1}^{K} \underbrace{q_k \overline{\theta}_k}_{MV^k}$$

- If market increases by x% and your market wealth increased by y% < x% then somebody else must have beaten the market!
- However, the **number** who beat can be smaller than the **number** who lose to the market. This happens when the small investors loose.
- Question: Is there persistence in who beats the market?

Some funds systematically outperform the market



No-Arbitrage Principle (4)

(6) There is a risk-return-decomposition:

$$E_{P_t}(\gamma_t^k) = r_t - \operatorname{cov}_{P_t}(\gamma_t^k, \ell_t)$$

where the **likelihood ratio process**: $\ell = \dfrac{\pi}{P}$

is also called *pricing kernel*

ideal security

and *stochastic discount factor*.

Summary: Equivalent are

- There is no arbitrage.
- Prices are π martingales.
- π expected discounted dividends model.
- Prices fluctuate π randomly.
- There are no π excess returns.
- There are no π expected gains.
- Risk is measured by the covariance to the likelihood ratio process $\frac{\pi}{P}$

References: Huang Litzenberger (1988): Foundations for Financial Economics.
LeRoy and Werner (2000): Principles in Financial Economics.

Overview of this course