University of Zurich
Institute for Empirical Research in Economics

Advanced Portfolio Theory

NHH-Bergen

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Contents

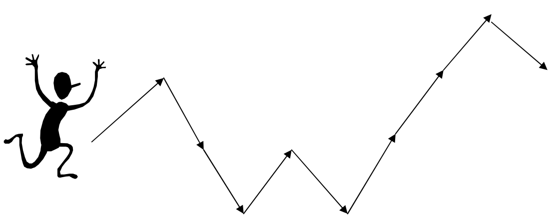
5. Chosing a Portfolio on a Random Walk: Diversification

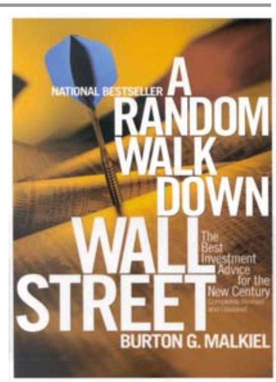
- Models of Random Walks
- Justifications of Random Walks
- Portfolio Choice

Random Walk Hypothesis

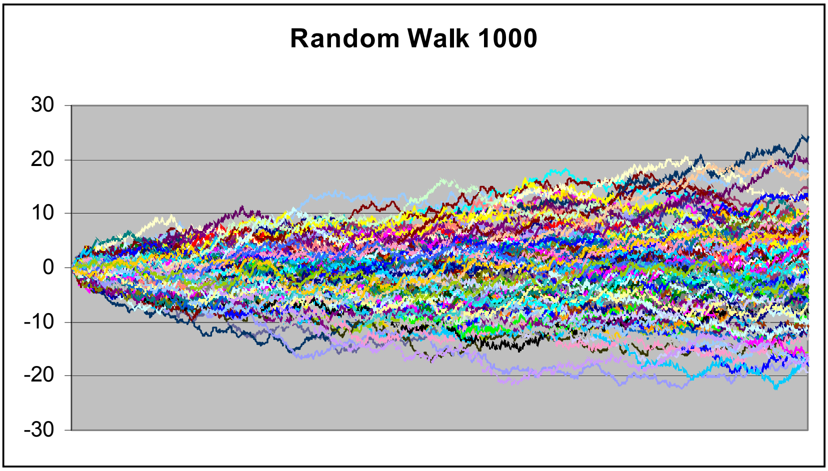
Stock market prices fluctuate

like a sequence of coin tosses.





Example of a Random Walk



a) Discrete Time Random Walk

additive: (prices) $q_{t+1}^{k} = q_{t}^{k} + \varepsilon_{t+1}$

multiplicative: (returns) $r_{t+1}^{k} = r_{t}^{k} + \varepsilon_{t+1}$ where $r_{t+1} = \ln(\frac{q_{t+1}}{q_t})$

where: ε_{t+1} $i.i.d$

$$E_{P_t}(\varepsilon_{t+1}) = 0, E_{P_t}(\varepsilon_{t+1}^2) = \sigma^2, E_{P_t}(\varepsilon_t \varepsilon_{t+1}) = 0$$

Gaussian RW: $\varepsilon_t \sim N(0,\sigma)$

Random walk with drift: $x_{t+1} = x_t + drift_t + \varepsilon_{t+1}$

a) Continuous Time Random Walk

Wiener Process: $d\ln q_t^k = \frac{dq_t^k}{q_t^k} = \mu\ dt + \sigma\ d\ W_t$

where:

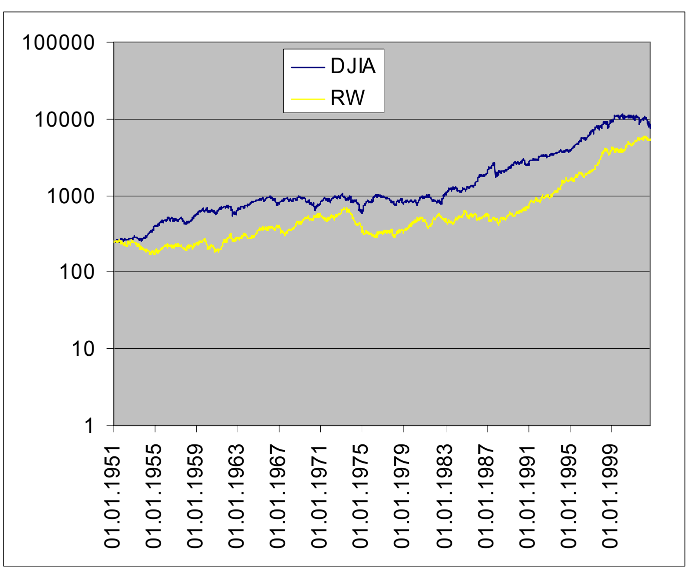
$W_0 = 0$

$W_t - W_\tau$ independently

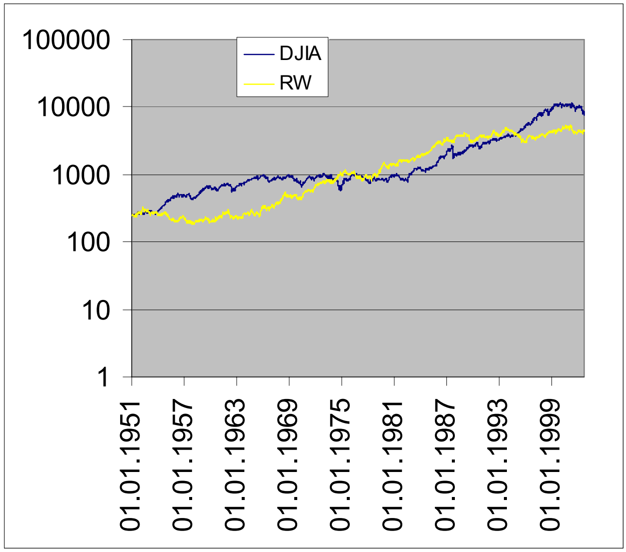
$W_t - W_\tau \sim N(\mu, t-\tau)$ distributed

$E_P[W_t] = 0\ ,\ E_P(W_t^{\ 2}) = t$

Bachelier Test: Which is the DJIA and which a RW?



Bachelier Test



b) Justification of RW-hypothesis

- Classical Finance:

  Anticipation Principle

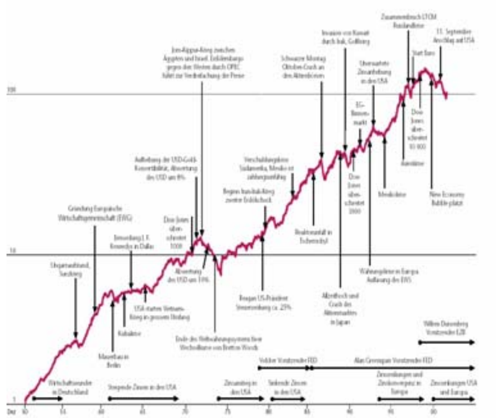
- Neoclassical Finance:

  No-Arbitrage Principle

Classical Finance

Anticipation principle:

- Every trend is already embodied in the prices.
- Stock market prices fluctuate according to the random arrival of new information.



$$q_{t+1} = E_t(q_{t+1}) + \varepsilon_{t+1}$$

$$\textit{Where} \quad \varepsilon_{t+1} = q_{t+1} - E_t(q_{t+1})$$

$$\textit{Hence} \quad E_t(\varepsilon_{t+1}) = 0$$

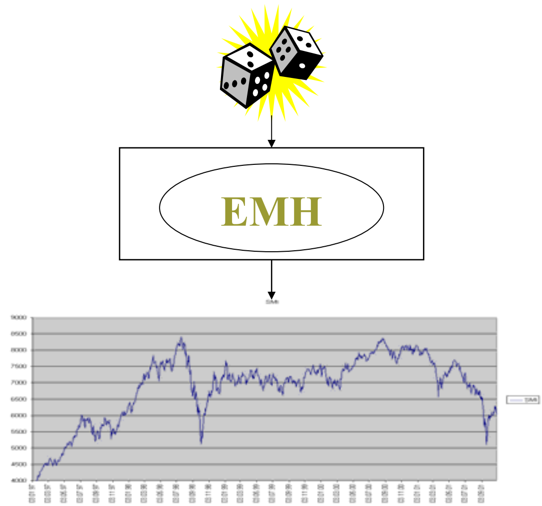
Anticipation Principle

„The only <u>price changes</u> that would occur are those that result from <u>new information.</u>

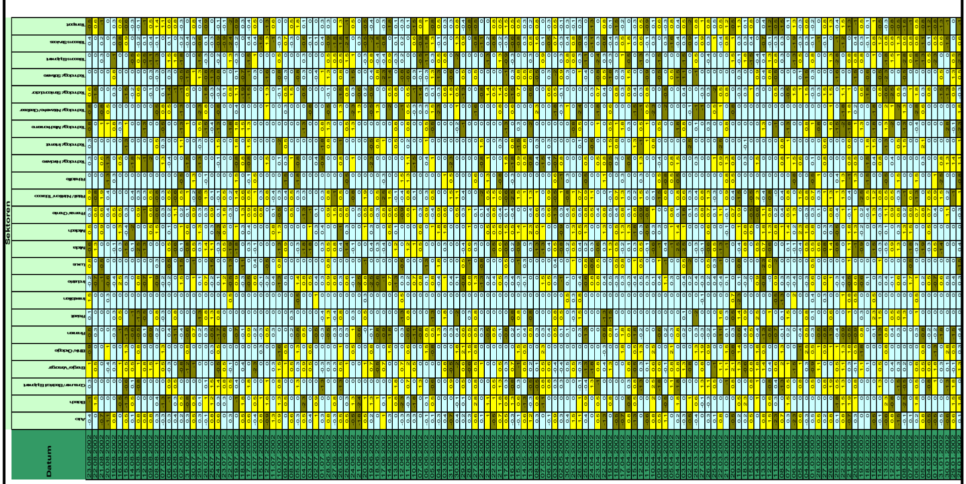
Since there is no reason to expect information to be non-random in appearance, the period-to-period price changes of a stock should be random movements, statistically independent of one another.“

Cootner (1964): *The Random character of stock prices*, MIT-press

Anticipation Principle



Case Study: The News Process as Perceived by Business Specialists



Stochastic Properties of News Process

- News are scaled from -3 to +3: ε_t
- Statistical Properties:
- E_t is independently and identically distributed over time
- $\mu(\varepsilon_t)=0$ for all t.

Conclusion: WHITE NOISE !

Neoclassical Finance

RW derived from No-arbitrage

- Deriving the π-RW Property
- Deriving the P-RW Property

Deriving the Π - RW Property

No-Arbitrage $\quad q_t^k = E^*_t(q^k_{t+1})$

Doob Decomposition: $\quad q^k_{t+1} = q^k_t + [q^k_{t+1} - E^*_t(q^k_{t+1})]$

$$q^k_{t+1} = q^k_t + \varepsilon_{t+1}$$

Hence: $\quad \varepsilon_{t+1} = q^k_{t+1} - E^*_t(q^k_{t+1})$

$$E^*_t[\varepsilon_{t+1}] = E^*_t[q^k_{t+1} - E^*_t(q^k_{t+1})] = 0$$

Where: $\quad \varepsilon_{t+1}$ is i.i.d. because "there is no reason to expect information to be non-random in appearance."

$E^*_t = 1/(1+rf) E_{\pi^*}$

Deriving the P- RW Property

No-Arbitrage $\quad q_t^k = E_t(q^k_{t+1}) + COV_t(q^k_{t+1}, l_{t+1})$

Doob Decomposition: $\quad q^k_{t+1} = q^k_t + [q^k_{t+1} - E_t(q^k_{t+1})] + Drift_{t+1}$

$$q^k_{t+1} = q^k_t + \varepsilon_{t+1} + Drift_{t+1}$$

Hence: $\quad \varepsilon_{t+1} = q^k_{t+1} - E_t(q^k_{t+1})$

$$E_t[\varepsilon_{t+1}] = E_t[q^k_{t+1} - E_t(q^k_{t+1})] = 0$$

Where: $\quad E_t = 1/(1+rf) E_P$

Note: Drift disappears for risk neutral market. We come back to risk aversion later.

c) Portfolio Choice

- CRRA and log normal returns
- Myopic portfolio choice
- On a RW myopic porftolio choice is fine

Campbell and Viceira (2002): Strategic Asset Allocation, chapter 2.

Constant Relative Risk Aversion

„The long-run behavior of the economy suggests that relative risk aversion cannot depend strongly on wealth. Per capita consumption and wealth have increased greatly over the past two centuries. Since financial risks are multiplicative, this means that the absolute scale of financial risks has also increased while the relative scale is unchanged. Interest rates and risk premia do not show any evidence of long-term trends in response to this long-term growth; this implies that investors are willling to pay almost the same relative costs to avoid given relative risks as they did when they were much poorer, which is possible only if relative risk aversion is almost independent of wealth.“

Campbell and Viceira (2002): Strategic Asset Allocation, chapter 2, page 24.

Log-normal Returns

- MV-Principle needs some normallity.
- Normally distributed returns are impossible:

1. prices do not become negative:

$$R_{t+1}^k = \frac{q_{t+1}^k}{q_t^k} \geq 0 \text{ and } R_{t+1}^k - 1 = \frac{q_{t+1}^k - q_t^k}{q_t^k} \geq -1$$

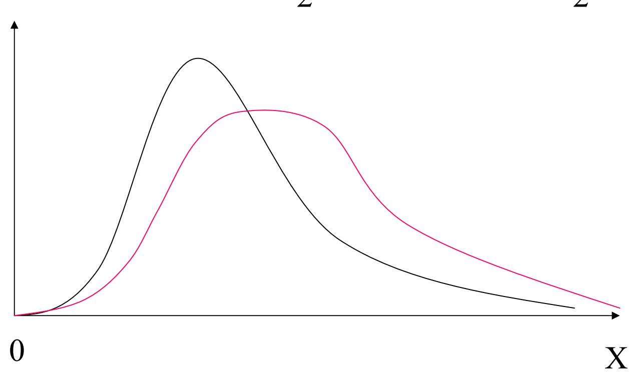
2. long run returns are products of short run returns and the product of normally distributed returns is not itself normally distributed

- Log-normally distributed gross returns r^k avoid these two problems.

A Fundamental Property of Log-Normal Variables

Let $\log(X) \sim N(\mu,\sigma)$ then

$$\log E(X) = E(\log X) + \frac{1}{2} VAR(\log X) = \mu + \frac{\sigma^2}{2}$$



Log-normal Returns

- Log-normal returns do not carry over to portfolio returns, because the sum of log-normal returns is not itself log-normal.
- This problem can be avoided considering short time intervals. In the limit of continuous time the problem is gone!

Campbell and Viceira (2002): Strategic Asset Allocation, chapter 2, page 25.

A Fundamental Property of Log-Normal Portfolios

Let there be k=1,...,K risky assets with $r_k = \log R_k$, $\{r_k\}_{k=1}^{K} \sim N(\mu, \sigma)$

and let $r_f = \log R_f$ be the log of the gross risk free rate (say k=0)

Consider the portfolio of risky assets $R_\lambda = \sum_{k=1}^{K} R_k \lambda_k$ with $\log R_\lambda = r_\lambda$.

Then we the approximation: $\mu(r_\lambda) \approx \sum_{k=1}^{K} \mu(r_k)\lambda_k + \frac{1}{2}\sum_{k=1}^{K}\sum_{j=1}^{K} \lambda_k \lambda_j \operatorname{cov}(r_k, r_j)$.

This approximation is better the smaller the time intervall.

In the limit for continuous time the approximation is exact.

Campbell and Viceira (2001): „Stock Market Mean Revision and the Optimal Equity Allocation of a Long-Lived Investor“, European Finance Review,??.

Myopic Portfolio Choice

… assuming CRRA and log-normal returns:

$$\max_{\lambda_t} E_t \frac{W_{t+1}^{1-\alpha}}{1-\alpha}$$

$$s.t. \quad W_{t+1} = R_{\lambda_t} W_t \text{ and } \sum_{k=0}^{K} \lambda_{k,t} = 1$$

$$where \quad R_{\lambda_t} = \sum_{k=0}^{K} \lambda_{k,t} R_{k,t+1}, \; with \quad R_{0,t+1} = R_{f,t}$$

Myopic Portfolio Choice

$$\max_{\lambda_t} E_t \frac{W_{t+1}^{1-\alpha}}{1-\alpha} \Leftrightarrow \max_{\lambda_t} \log \left[E_t \frac{W_{t+1}^{1-\alpha}}{1-\alpha} \right]$$

$$\Leftrightarrow \max_{\lambda_t} \log E_t W_{t+1}^{1-\alpha} - \log(1-\alpha)$$

$$\Leftrightarrow \max_{\lambda_t} (1-\alpha) E_t \log W_{t+1} + \frac{(1-a)^2}{2} VAR(\log W_{t+1})$$

$$\Leftrightarrow \max_{\lambda_t} (1-\alpha) E_t r_{\lambda_t} + \frac{(1-a)^2}{2} \sigma_{r_{\lambda_t}}^2$$

$$\Leftrightarrow \max_{\lambda_t} E_t r_{\lambda_t} + \frac{(1-a)}{2} \sigma_{r_{\lambda_t}}^2$$

$$\Leftrightarrow \max_{\lambda_t} \sum_{k=1}^{K} \lambda_{k,t} (E_t r_{k,t+1} - r_{f,t}) + \frac{a}{2} \sum_{k=1}^{K} \lambda_{k,t} \sigma_{r_{k,t+1}}^2 - \frac{a}{2} \sum_{k=1}^{K} \sum_{j=1}^{K} \lambda_{k,t} \lambda_{j,t} \operatorname{cov}(r_{k,t+1}, r_{j,t+1})$$

Solution to Myopic Portfolio Choice

$$\lambda_1^{opt} = \frac{1}{\alpha^i}\operatorname{cov}_t^{-1}\left(E_t(r_{t+1}) - r_{f,t}\begin{pmatrix}1\\ .\\ 1\end{pmatrix} + \frac{1}{2}\sigma_t^2 \right)$$

where r_{t+1} is the vector of asset log-returns

and I is the vector with all entries 1,

where σ_t^2 is the vector of variances of the k assets.

Note that except for $+\frac{1}{2}\sigma_t^2$ this is the same solution as always!

Dynamic Portfolio Choice with CRRA, Lognormal i.i.d. Returns

Samuelson-Merton-Theorem: No-Time Diversification

Consider the n-period portfolio problem given the evolution of wealth

$\mathrm{W}_{\mathrm{t+n}} = R_{t+n}R_{t+n-1}R_{t+n-2}...R_t W_t$ taking logs we get:

$w_{\mathrm{t+n}} = r_{t+n} + r_{t+n-1} + r_{t+n-2} ... + r_t + w_t$

Suppose the log-normal returns are i.i.d..Then we get:

$E_{t+n}(w_{\mathrm{t+n}}) = nE_t(w_{\mathrm{t+1}})$ and also $VAR_{t+n}(w_{\mathrm{t+n}}) = nVAR_t(w_{\mathrm{t+1}})$.

Hence the n-period portfolio problem is only a scaled up one period problem!

Thus if investors have CRRA then the asset allocation does not depend on wealth and since expectations and covariances are constant over time the investor always chooses the same portfolio: The myopic solution is also the dynamic solution!

Samuelson (1969):"Lifetime Portfolio Selection by Dynamic Stochastic Programming", Review of Economics and Statistics (51), pp.239-246.

Rebalancing, Fix Mix and Volatility Pumping (I)

The "No Time Diversification Theorem" of Samuelson and Merton shows that an on an i.i.d. process with log-normal returns an expected utility maximizer with CRRA chooses an asset allocation that is invariant over time. Hence when the price of an asset goes up, the investor sells part of his holdings of that asset and when the price goes down he will purchase more of this asset in order to keep the share of wealth invested in this asset fixed over time. This strategy is also called "fix mix" and the corresponding behavior is called "rebalancing".

The next example, "volatility pumping", illustrates that in a sense a rebalancing behavior is more successful the more volatile the assets are.

Luenberger (1998): Investment Science, Chapter 15.

Rebalancing, Fix Mix and Volatility Pumping (II)

Volatility pumping example:

There are two asset, a stock that in each period has a 50:50 chance of either doubling its value or of reducing it by one-half and cash. The stochastic process is i.i.d. as for example a repeated coin tossing.

An investment left in the stock will have a value that fluctuates a lot but has no overall growth rate. Also the investment in cash has no growth rate. However if we rebalance, say a (½, ½) asset allocation our wealth grows at a rate of about 6%:

$$\mu(g) = \frac{1}{2}\ln(\frac{1}{2}+1) + \frac{1}{2}\ln(\frac{1}{2}+\frac{1}{4}) \approx 0.059$$

Rebalancing, Fix Mix and Volatility Pumping (III)

Let $\omega_t \in \{H, T\}$ be state of the world in period t.

And let $\omega^t = (\omega_0, \omega_1, ..., \omega_t)$ be the path up to period t.

Then the evolution of wealth is given by the recursion:

$$w(\omega^{t+1}) = \left[A(\omega^{t+1})\lambda + (1-\lambda)\right] w(\omega^t)$$

where $A(\omega^{t+1})$ is 2 (in the case of H) or $\frac{1}{2}$(in the case of T).

Since ω^t is i.i.d. $A(\omega^{t+1}) = A(\omega_t)$. By the Law of Large Numbers we get for the expected evolution of log-returns:

$$E\ln(w(\omega^{t+1})) = p(H)\ln\left[2\lambda + (1-\lambda)\right] + p(T)\ln\left[\frac{1}{2}\lambda + (1-\lambda)\right] + \ln w(\omega_0)$$

Rebalancing, Fix Mix and Volatility Pumping (IV)

Hence the expected growth rate is $\mu(\text{g}(\lambda))=\frac{1}{2}\ln\left[2\lambda + (1-\lambda)\right] + \frac{1}{2}\ln\left[\frac{1}{2}\lambda + (1-\lambda)\right]$.

We see that, as claimed above the expected growth rate of putting all money into one of the two assets, be it the risky or the risk-free asset, is zero:

$$\mu(\text{g}(1))=\frac{1}{2}\ln[2] + \frac{1}{2}\ln\left[\frac{1}{2}\right] = \mu(\text{g}(0))=\frac{1}{2}\ln[1] + \frac{1}{2}\ln[1] = 0.$$

*H*owever, rebalancing the portfolio $(\lambda, 1\text{-}\lambda) = \left(\frac{1}{2}, \frac{1}{2}\right)$ gives:

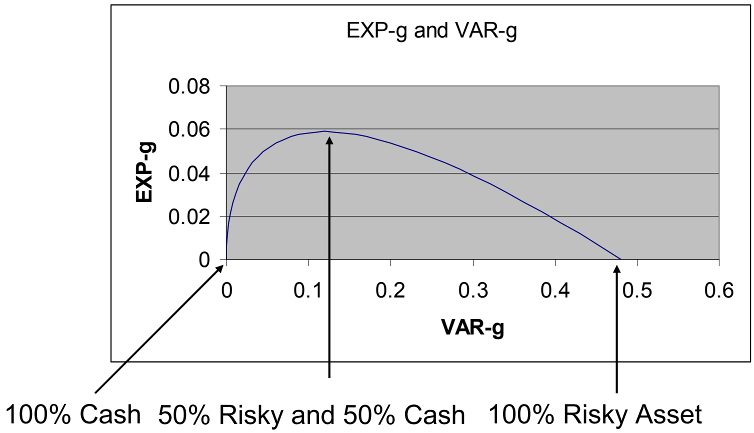
$$\mu(\text{g}\left(\frac{1}{2}\right)) = \frac{1}{2}\ln\left[\frac{1}{2}+1\right] + \frac{1}{2}\ln\left[\frac{1}{2}+\frac{1}{4}\right] \approx 0.059.$$

Variances of expected growth rates:

$$\sigma^2(\text{g}(\lambda))=\frac{1}{2}\left[\ln\left[2\lambda + (1-\lambda)\right] - \mu(\text{g}(\lambda))\right]^2 + \frac{1}{2}\left[\ln\left[\frac{1}{2}\lambda + (1-\lambda)\right] - \mu(\text{g}(\lambda))\right]^2.$$

Rebalancing, Fix Mix and Volatility Pumping (V)

Mean-Variance of growth rates:



Rational or Irrational?

"I believe that the market is efficient therefore I follow a passive strategy and simply buy and hold the market portfolio."

Compatibility with asset market equilibrium (I)

We have seen:

If log-returns are i.i.d. and Gaussian then all CRRA-Eu-maximizers choose a fix-mix strategy with the same fund of risky assets λ1.

Why then should asset prices fluctuate at all? I.e. can this portfolio choice be compatible with an asset pricing model?

Consider asset market equilibrium:

Demand of investor i at period t: $\theta_t^{i,k} = \frac{\lambda_t^{i,k}}{q_t^k} w_t^i$.

Market clearing: $\underbrace{1}_{\text{supply}} = \underbrace{\sum_i \theta_t^{i,k}}_{demand}$. (Supply is normalized to 1)

Compatibility with asset market equilibrium (II)

This gives: $q_t^k = \sum_i \lambda_t^{i,k} w_t^i$. (Price of asset k is the wealth average of the strategies.)

With a common and stationary strategy λ this gives:

$$q_t^k = \lambda^k \sum_i \frac{1}{\alpha^i} w_t^i = \lambda^k \frac{1}{\alpha^{rep}} \sum_i w_t^i$$ (for a representative consumer)

How is aggregate wealth determined in this model?

By aggregate dividends: $\sum_i w_t^i = \sum_k D_t^k$.

Hence prices fluctuate with aggregate dividends.

I.e. if aggregate dividends are i.i.d. (and Gaussian)

then prices are i.i.d and Gaussian.

Dynamic Portfolio Choice

$$\text{Consider } \max E_t \sum_{\tau=0}^{\infty} \delta^{\tau} u(C_{t+\tau})$$

$$\text{s.t. } W_{t+1} = R_{\lambda,t+1}(W_t - C_t)$$

$$\textit{where } \; R_{\lambda,t+1} = \sum_k R_{k,t+1}\lambda_k$$

$$\textit{Euler equation}: \; u'(C_t) = E_t\left[\delta u'(C_{t+1})R_{k,t+1}\right], k = 1,\ldots,K.$$

$$\textit{Define the Stochastic Discount Factor as } SDF_{t+1} = \delta \frac{u'(C_{t+1})}{u'(C_t)}$$

$$\textit{Then } \; 1 = E_t\left[SDF_{t+1}R_{k,t+1}\right], k = 1,\ldots,K$$

$$\textit{in particular } \; \frac{1}{R_{f,t+1}} = E_t\left[SDF_{t+1}\right]$$

where $R_{f,t+1}$ is the gross risk free rate from t to t+1.

Log-normal Returns and CRRA

$$\textit{Assume } \; E_t \sum_{\tau=0}^{\infty} \delta^{\tau} u(C_{t+\tau}) = E_t \sum_{\tau=0}^{\infty} \delta^{\tau} \frac{C_{t+\tau}^{1-\alpha} - 1}{1-\alpha} \text{ and } \{r_k = \log R_k\}_{k=1}^{K} \sim N(\mu, \sigma)$$

Then denoting $\Delta c_{t+1} = \log C_{t+1} - \log C_t$ (consumption growth),
the Euler equation becomes:

$$E_t(\Delta c_{t+1}) = \frac{\log \delta}{\alpha} + \frac{1}{\alpha} r_{f,t+1} + \frac{\alpha}{2}\sigma_{c_{t+1}}^2$$

Discount factor

Consumption risk

Risk preference

Rate at which consumption
increases from t to t+1

Hansen and Singleton (1983):"Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns", Journal of Political Economy (91), pp.249-268.

Intertemporal CAPM

For Log-normal Returns and CRRA we can derive,

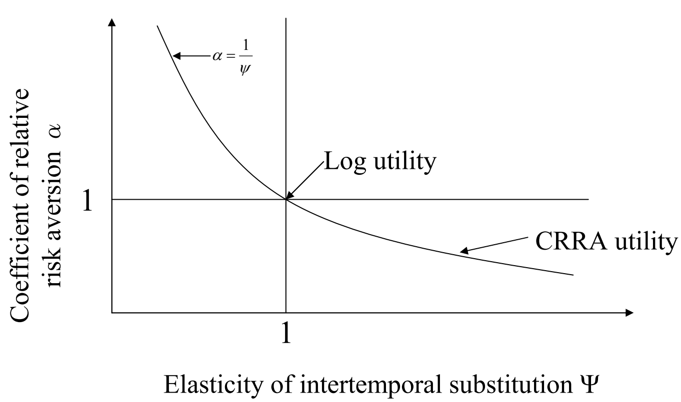
using the same ideas as above:

$$E_t r_{k,t+1} - r_{f,t+1} + \frac{\sigma_{k,t}^2}{2} = \alpha \operatorname{cov}_t(r_{k,t+1,} \Delta c_{t+1}), k = 1\ldots, K$$

Breeden (1979): „An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities“,Journal of Financial Economics (7), pp. 265-296.

Epstein and Zin Representation of Eu

$$U_t(C) = \left[(1-\delta) C_t^{\frac{(1-\alpha)}{\theta}} + \delta (E_t U_{t+1}^{1-\alpha})^{\frac{1}{\theta}} \right]^{\frac{\theta}{(1-\alpha)}} \quad \text{where } \theta \equiv \frac{(1-\alpha)}{(1-\frac{1}{\psi})}$$



Intertemporal CAPM

For Log-normal Returns and CRRA we can derive,

using the same ideas as above:

$$E_t r_{k,t+1} - r_{f,t+1} + \frac{\sigma_{k,t}^2}{2} = \alpha \operatorname{cov}_t(r_{k,t+1,} \Delta c_{t+1}), k = 1 \ldots, K$$

For Log-normal Returns and Epstein Zin Utility we can derive, using the same ideas as above:

$$E_t r_{k,t+1} - r_{f,t+1} + \frac{\sigma_{k,t}^2}{2} = \theta \frac{\operatorname{cov}_t(r_{k,t+1}, \Delta c_{t+1})}{\psi} + (1-\theta)\operatorname{cov}_t(r_{k,t+1}, r_{\lambda,t+1}), k = 1 \ldots, K$$

Campbell and Viceira (2002): Strategic Asset Allocation, OUP,Chapter 2.