Markets Do Not Select For a Liquidity Preference as Behavior Towards Risk*

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May 16, 2004

Abstract

Tobin (1958) has argued that in the face of potential capital losses on bonds it is reasonable to hold cash as a means to transfer wealth over time. It is shown that this assertion cannot be sustained focusing on the evolution of wealth of cash holders versus non-cash holders. Cash holders will be driven out of the market in the long run by traders who only use a (risky) long-lived asset to transfer wealth. Similarly, in a model with a bond instead of cash, depending on the way consumption is modeled, bond holders do not survive in the presence of pure stock holders.

JEL-Classification: G11, E41, D81
Keywords: evolutionary finance, portfolio theory, demand for money.

*We are grateful to Enrico De Giorgi and two anonymous referees for valuable comments. Financial support by the national centre of competence in research "Financial Valuation and Risk Management" is gratefully acknowledged. The national centers in research are managed by the Swiss National Science Foundation on behalf of the federal authorities.

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1 Introduction

Using two-period mean-variance analysis, Tobin (1958) has argued that in the face of potential capital losses on bonds it is reasonable to hold cash as a means to transfer wealth over time. He concludes on page 66: *"If cash is to have any part in the composition of investment balances, it must be because of expectations or fear of loss on other assets."* While this assertion is certainly true for two-period models, we argue here that it cannot be sustained when focusing on the long run evolution of the wealth distribution. Analyzing the wealth dynamics resulting from cash and asset holdings, we show that cash holders will be driven out of the market by traders who only use a (risky) long-lived asset to transfer wealth.

The main idea of this point is rather simple. The zero return on cash can dominate the return on any other asset with non-negative payoffs only if this asset generates capital losses. That is to say, only if the price of the other asset decreases. The price of an asset having non-negative payoffs does however not decrease below zero. Therefore capital losses are bounded and eventually the rate of return on the asset will dominate the zero rate of return on money.

This observation is similar to the one made by Hellwig (1993). He argues that Tobin (1958)'s assertion cannot be sustained in an infinite horizon model with rational expectations. As Hellwig emphasizes, money can have a positive value in any point in time only if some agents believe that the other assets may have sufficiently severe capital losses in the future. Hence rational agents must expect a non-ending sequence of severe capital losses which is inconsistent with positive asset prices. Hellwig (1993)'s argument does however leave open the possibility that holding cash can be sustained with other types of expectations like rational beliefs, Kurz (1994), for example.

We show that the general idea of the inferiority of holding cash in the long run does not need to be based on the assumption of rational expectations. Actually, the notion of expectations is not necessary at all to make this point. Neither is there a need to evoke any demanding equilibrium concept based on rational expectations or other types of expectation formation. We show that the wealth of all cash holders will eventually become a negligible part of total wealth in the market. Hence using the evolution of wealth as the criterion one can argue that cash holding is an inferior strategy.

The point made in our paper can be extended to evolutionary finance models with a risky and a risk-free asset which is not cash but a bond with positive interest. Those models can be found in e.g. Arthur, Holland, LeBaron, Palmer, and Taylor (1997), LeBaron, Arthur, and Palmer (1999), Brock and Hommes (1998), LeBaron (2001) and Lux (1998), among others. The first observation in this setting is that with a positive interest rate and a relatively small dividend, capital losses on the risky asset can result in a return dominance of the risk-free asset.

Hence the straightforward argument showing that the risk-free asset will eventually be dominated by the risky asset is no longer valid. However, if, as in the case of cash, the risk-free asset is also the numeraire in which payoffs are delivered, a different argument can be made. The higher the returns the more affluent the numeraire and hence the more likely there will be capital gains on the risky asset and again the risk-free asset is dominated. One way of avoiding the dominance of the risky asset is to introduce consumption that reduces the amount of the numeraire in the economy. Indeed if, as in the famous Lucas (1978) model, all payoffs are in terms of a single perishable consumption good then by construction the consumption is always equal to the inflow of returns. Evolutionary finance models based on Lucas (1978) are e.g. considered in Blume and Easley (1992, 2001), Sandroni (2000), and Evstigneev, Hens, and Schenk-Hoppé (2003). If, however, one does not follow the Lucas (1978) construction and uses the more standard evolutionary finance approach in which the risk-free asset is a long-lived bond, then the potential dominance of non-bond holders depends on the specific way one models consumption. If consumption is proportional to the dividends or proportional to the interest earned, non-bond holders will dominate. However, if, as e.g. in LeBaron (2001), consumption is proportional to wealth then this is not necessarily the case. An alternative way to avoid the clear dominance of the risky asset is to assume, as e.g. in Brock and Hommes (1998), that the risky asset is in zero net supply. If investors are myopic mean-variance maximizers, the demand for the risky asset is independent of the wealth level and investors hold ever increasing amounts of the risk-free asset.

We illustrate our point in a simple model with two long-lived assets in positive net supply. One riskless asset, whose price will be chosen as the numeraire, and one risky asset that is risky both in terms of dividends and resale value. Both assets' payoffs are denominated in terms of the numeraire which might also be used for consumption. Hence the payoffs of the assets are in terms of a storable asset and depending on the degree of consumption the amount of the numeraire in the model grows over time.

Various interpretations of this simple setting are possible. In the case the return on the riskless asset is zero, one may consider the riskless asset as money/cash and one could think of the risky asset as being a console/bond. This is exactly the setting of Tobin (1958). In case the return on the riskless asset is positive, one can consider the riskless asset to be a bond while the risky asset may be interpreted as a stock. This is done in many evolutionary finance models, e.g. Arthur, Holland, LeBaron, Palmer, and Taylor (1997), LeBaron, Arthur, and Palmer (1999), Brock and Hommes (1998), LeBaron (2001) and Lux (1998), among others.

The next section sets up the model, then we derive our main result and finally we conclude.

2 The Model

Time is discrete and denoted by $t = 0, 1, 2, ...$ There are two long-lived assets, one risky and one risk-free. The risk-free asset is used as the numeraire. Payoffs of both assets are in terms of the numeraire. The risk-free asset pays gross interest at a rate $R = (1 + r) \geq 0$. Note that the other asset may be risky both in terms of dividends and resale value. In every period this asset pays off a dividend $D_t(s^t) \geq 0$, where $s^t = (s_0, ..., s_t)$ is the history of states of the world s_u, $u \leq t$, up to period t. The risk-free asset can be used for consumption.

There are $I \geq 2$ agents who can hold both assets to transfer wealth across time. Investor i's portfolio contains m_t^i units of the risky asset and a_t^i units of the long-lived asset. As in Tobin (1958) we assume that short selling is not possible. This will in particular rule out negative price bubbles. Alternatively we could have introduced short selling bounded by some arbitrary lower limit.

It is not essential for our reasoning, how the demands m_t^i and a_t^i are determined. They could stem from completely rational agents maximizing expected utility over the infinite horizon, or boundedly rational agents solving myopic two period maximization problems. It is even allowed to dismiss any rationality interpretation. In this note we consider the evolution of wealth for any sequence of demands m_t^i and a_t^i.

Investor i's wealth in period t after dividend payment is given by

$$w_t^i = R\, m_{t-1}^i + (D_t(s^t) + q_t)\, a_{t-1}^i \qquad (1)$$

where q_t denotes the price of the asset in terms of the numeraire asset. The budget restriction is

$$m_t^i + q_t\, a_t^i + c_t^i = R\, m_{t-1}^i + (D_t(s^t) + q_t)\, a_{t-1}^i \qquad (2)$$

where for some interpretations of the model the variable c_t^i (which can depend on s^t) can be used to model agent i's consumption in period t.

The risky asset is in fixed positive supply (normalized to one), while the supply of the numeraire asset is endogenously given by the cumulated compounded dividends exceeding $C_t = \sum_{i=1}^{I} c_t^i$. Thus the market clearing conditions are given by

$$\sum_{i=1}^{I} a_t^i = 1 \;\text{ and }\; \sum_{i=1}^{I} m_t^i = \sum_{\tau=0}^{t} R^{t-\tau}\, (D_\tau(s^\tau) - C_\tau) \;\text{ for all } t = 0, 1, ... \qquad (3)$$

Walras' law ensures that the assumption of infinite elastic supply of the numeraire asset (which is often made) does not change this setup. Summing all budget constraints (2) over investors and assuming market clearing for the risky asset one obtains

$$\sum_{i=1}^{I} m_t^i + \sum_{i=1}^{I} c_t^i = R \sum_{i=1}^{I} m_{t-1}^i + D_t(s_t).$$

Substitution forwards from $\sum_{i=1}^{I} m_0^i = D_0 - C_0$ gives the market clearing condition for the numeraire asset.

This model embodies at least the following four important cases. When $R = 1$, the riskless asset is "cash" with net interest rate $r = 0$. An infinite time-horizon version of Tobin (1958)'s model is then obtained in the absence of consumption, i.e. $c_t^i \equiv 0$. When $R = 0$, the riskless asset is a perishable consumption good as in Lucas (1978). In this case consumption can fully be accounted for by m_t^i and one can set the variable c_t^i to zero. The numeraire asset is completely used for consumption and current market demand $\sum_{i=1}^{I} m_t^i$ equals current dividends. Wealth is only transferred with the risky asset. Two other specialized versions of the model are also of interest. Consumption in both cases can be accommodated while setting the variable c_t^i to zero. First, if investors consume proportional to their dividend income, $c_t^i = (1 - \beta) D_t(s^t) a_{t-1}^i$, the budget constraint (2) shows that one can equivalently set the consumption variable c_t^i to zero and replace the dividend process by $\beta D_t(s^t)$. Second, when consumption is proportional to earned interest income, $c_t^i = (1 - \beta) R m_{t-1}^i$, one can set the consumption variable c_t^i to zero and replace the interest rate by βR. Of course one can also combine the last two cases—even allowing for different coefficients β—and assume that investors consume proportionally to both incomes earned. Shefrin and Statman (1984) have shown that consumption proportional to interest and/or dividend earnings are commonly observed and can be justified by adhering to simple heuristics like "Do not dig into capital." Those rules help to avoid self-control problems.

In the following we will focus on the model with $c_t^i = 0$ that includes all the versions discussed above. Note that, as argued above, in all but the first of these versions this does not prevent an interpretation of the model including consumption. The main result in this paper deals with the case $c_t^i = 0$. At the end of the next section we comment on the case in which consumption is proportional to wealth, $c_t^i = (1 - \beta) w_t^i$, as considered e.g. in LeBaron (2001) where β is a discount factor given by some intertemporal expected logarithmic utility function.

For $c_t^i = 0$ the budget restriction of investor i reads

$$m_t^i + q_t\, a_t^i = R\, m_{t-1}^i + (D_t(s^t) + q_t)\, a_{t-1}^i \tag{4}$$

Considering the right-hand side of this budget constraint, we can already make the intuition of the general argument outlined in the introduction more precise.

In the case of $R \leq 1$, which for example holds with cash ($r = 0$), we obtain that whenever the risky asset has some positive payoff, its return dominates that of the riskless asset, provided there are no capital losses. Since capital losses are bounded and since the horizon of the model is infinite, eventually holding the numeraire asset will then be dominated. With a net return $R > 1$ on the riskless asset, the intuition for our result becomes clear, once the formation of prices has been explained. As we show below, prices increase with market wealth. Hence, the more returns and dividends are paid, the more likely there are capital gains on the risky asset.

Holdings of agents are described in terms of budget shares. Let λ_t^i denote the fraction of wealth an investor i assigns to the purchase of the asset and by $1 - \lambda_t^i$ the fraction of wealth assigned to holding the numeraire asset, i.e.

$$m_t^i = (1 - \lambda_t^i)\, w_t^i \quad \text{and} \quad a_t^i = \frac{\lambda_t^i\, w_t^i}{q_t} \tag{5}$$

Rewriting (1) one obtains

$$w_t^i = R\,(1 - \lambda_{t-1}^i)\, w_{t-1}^i + (D_t(s_t) + q_t)\, \frac{\lambda_{t-1}^i\, w_{t-1}^i}{q_{t-1}} \tag{6}$$

Equation (5) implies that the market-clearing price q_t is given by

$$q_t = \sum_i \lambda_t^i\, w_t^i = \lambda_t\, w_t \tag{7}$$

where $\lambda_t = (\lambda_t^1, ..., \lambda_t^I) \geq 0$ and $w_t^T = (w_t^1, ..., w_t^I) \geq 0$. If for some investor $\lambda_t^i\, w_t^i > 0$, then $q_t > 0$. Thus prices are positive as long as some investor with positive wealth demands a positive fraction of the risky asset. Since wealth increases with returns both on the riskless and on the risky asset, prices are more likely to rise when the riskless asset has positive returns. Hence in that case capital losses are even less likely than in the case of money/cash.

Inserting (7) in (6) yields an implicit equation for the wealth of investor i in period t for each given distribution of wealth across investors w_{t-1} in period $t-1$. Define

$$A_{t-1}^i = R\,(1 - \lambda_{t-1}^i)\, w_{t-1}^i + D_t(s^t)\, B_{t-1}^i$$

and

$$B_{t-1}^i = \frac{\lambda_{t-1}^i\, w_{t-1}^i}{\lambda_{t-1}\, w_{t-1}}$$

where the subscript $t-1$ refers to the time-dependence of the wealth distribution w_{t-1}. This implicit equation for the evolution of wealth can then be written as

$$w_t = A_{t-1} + B_{t-1}\, \lambda_t\, w_t \tag{8}$$

with $A_{t-1}^T = (A_{t-1}^1, ..., A_{t-1}^I)$ and $B_{t-1}^T = (B_{t-1}^1, ..., B_{t-1}^I)$. One needs to solve (8) for w_t to derive the law of motion for the distribution of wealth across investors.

From (8) we obtain

$$w_t = (I - B_{t-1}\,\lambda_t)^{-1}\,A_{t-1} \tag{9}$$

where I is the identity matrix in $\mathbb{R}^I$. The inverse of $I - B_{t-1}\,\lambda_t$ is given by $I + (1-\lambda_t\,B_{t-1})^{-1}\,B_{t-1}\,\lambda_t$ provided $\lambda_t\,B_{t-1} \neq 1$ (Horn and Johnson 1985, Sec. 0.7.4). It is straightforward to check that in our model $\lambda_t\,B_{t-1} < 1$, if for some investor $\lambda_t^i < 1$ and $\lambda_{t-1}^i\,w_{t-1}^i > 0$.

One finally obtains

$$w_t = \left(I + \frac{1}{1-\lambda_t\,B_{t-1}}\,B_{t-1}\,\lambda_t\right) A_{t-1} \tag{10}$$

where the ith component of (10) is given by

$$\begin{aligned} w_t^i \quad = \quad & \Bigg(R\,(1-\lambda_{t-1}^i) + D_t(s^t)\,\frac{\lambda_{t-1}^i}{\lambda_{t-1}\,w_{t-1}} \\ & + \lambda_{t-1}^i\,\frac{\sum_j \left[R\,(1-\lambda_{t-1}^j) + D_t(s^t)\,\frac{\lambda_{t-1}^j}{\lambda_{t-1}\,w_{t-1}} \right] \lambda_t^j w_{t-1}^j}{\sum_j (1-\lambda_t^j)\lambda_{t-1}^j w_{t-1}^j} \Bigg)\, w_{t-1}^i \end{aligned}$$

It is clear from the above discussion and (10) that the evolution of the wealth distribution is well-defined if at least one investor i with initial wealth $w_0^i > 0$ adopts an investment rule with $\lambda_t^i \in (0,1)$ for all t.

3 The Main Result

Tobin (1958)'s justification of holding cash in the presence of potential capital losses is now addressed in the model introduced above for the particular case of two investors. The first investor only holds the risky asset to transfer wealth across time ($\lambda_t^1 = 1$) while the second investor holds a mixed portfolio and invests partly in the numeraire asset ($0 < \lambda_t^2 < 1$). Both investors are endowed with initial wealth $w_0^i > 0$. Under these two assumptions the equation governing the evolution of wealth (10) is well-defined. The model with $I = 2$ turns out to be analytically tractable because the inverse matrix in (10) has a simple expression. In particular one can study the long-run distribution of wealth in this case.

After some lengthy but elementary calculations (see the appendix) one obtains that for $I = 2$, (10) is equivalent to

$$w_t^1 \quad = \quad \frac{D_t(s^t) + R\,(1-\lambda_{t-1}^2)\,\lambda_t^2\,w_{t-1}^2}{(1-\lambda_t^2)\,\lambda_{t-1}^2\,w_{t-1}^2}\,w_{t-1}^1$$

$$w_t^2 = \frac{D_t(s^t) + R\,(1-\lambda_{t-1}^2)\,w_{t-1}^2}{(1-\lambda_t^2)\,w_{t-1}^2}\,w_{t-1}^2 \tag{11}$$

From equations (11) we can analyze the possibility of capital losses on the risky asset. To this end compute its price change, which is, by (7), $q_t - q_{t-1} = w_t^1 + \lambda_t^2 w_t^2 - (w_{t-1}^1 + \lambda_{t-1}^2 w_{t-1}^2)$. Inserting (11) one obtains

$$\begin{aligned} q_t - q_{t-1} &= \frac{D_t(s^t) + \left[R(1-\lambda_{t-1}^2)\lambda_t^2 - (1-\lambda_t^2)\lambda_{t-1}^2\right]\,w_{t-1}^2}{(1-\lambda_t^2)\,\lambda_{t-1}^2\,w_{t-1}^2}\,w_{t-1}^1 \\ &+ \frac{\lambda_t^2\,D_t(s^t) + \left[R(1-\lambda_{t-1}^2)\lambda_t^2 - (1-\lambda_t^2)\lambda_{t-1}^2\right]\,w_{t-1}^2}{(1-\lambda_t^2)\,w_{t-1}^2}\,w_{t-1}^2 \end{aligned} \tag{12}$$

If dividends on the risky asset were negligible, capital losses would occur if and only if

$$R < \frac{1-\lambda_t^2}{\lambda_t^2}\frac{\lambda_{t-1}^2}{1-\lambda_{t-1}^2}$$

This is true for instance if the second agent's budget share for the risky asset λ_t^2 strongly decreases. We see from this expression that capital losses are the more likely the smaller the risk-free return R. In particular when $\lambda_t^2 = 1$ then the condition reduces to $R < 1$. However these losses are bounded from below because prices cannot become negative.

With $R \geq 1$, the total wealth $W_t = w_t^1 + w_t^2$ of the economy may become arbitrarily large as time tends to infinity. In this case the dividends may become negligible in the long-run. For instance, it is apparent from (3) and (4) that the aggregate wealth tends to infinity (almost surely) if dividend payments $\sum_{\tau=0}^t D_\tau(s^\tau) \to \infty$ (almost surely): $W_t \geq m_t^1 + m_t^2 = \sum_{\tau=0}^t R^{t-\tau}\,D_\tau(s^\tau) \geq \sum_{\tau=0}^t D_\tau(s^\tau)$, if $R \geq 1$. Hence, if dividends are uniformly bounded from above, the dividend-wealth ratio, $D_t(s^t)/W_t$, converges to zero and the risk in the risky asset become negligible.

Of course in reality this problem does not occur because the ratio of dividends to nominal GDP has no trend. On US Data from 1981 to 2001, for example, this ratio fluctuates between 1% and 0.7% without any clear trend[1]. Therefore we make the following assumption:

(A) $D_t(s^t) = d(s_t)\,W_{t-1}$ *and* $d(s_t) \geq 0$ *is an ergodic process such that* $d(s) > 0$ *with positive probability.*

That is, the dividend grows on average with the same rate as the economy.

[1] See the chart on our web page: www.evolutionaryfinance.ch

Under assumption (A) one obtains from (11) an equation for the evolution of the ratio of the investors' wealth shares $r_t^i = w_t^i / W_t$:

$$\left(\frac{w_t^1}{w_t^2} = \right) \; \frac{r_t^1}{r_t^2} = \frac{1}{\lambda_{t-1}^2} \; \frac{d(s_t) + R\,(1-\lambda_{t-1}^2)\,\lambda_t^2\,r_{t-1}^2}{d(s_t) \; + R\,(1-\lambda_{t-1}^2)\,r_{t-1}^2} \cdot \frac{r_{t-1}^1}{r_{t-1}^2} \tag{13}$$

We finally assume

(B) There is a $\delta > 0$ such that $\delta \leq \lambda_t^2 \leq 1 - \delta$ for all t.

Under this assumption investor 2 cannot mimic investor 1's investment strategy who only holds the risky asset. In fact it suffices to require that the fraction of wealth allocated to the purchase of the risky asset by investor 2 does not tend to zero or to one too fast.

Theorem 1 *Under assumptions (A) and (B) the investor holding only the risky asset (while the other investor also holds the numeraire asset) gathers total wealth almost surely. The investor with a mixed portfolio becomes extinct.*

From an evolutionary perspective, investors should only hold the risky asset. Moreover, the result ensures that if, as in Lucas (1978), asset payoffs are denominated in terms of a perishable consumption good (i.e. $R = 0$), an agent who does not consume will (almost surely) dominate any agent that uses some of his wealth for consumption. This suggests that it is legitimate to assume a common consumption rate for all agents when analyzing what is the best allocation of wealth among *several* risky assets in an evolutionary finance model.

Proof of Theorem 1 The main task in this proof is to derive a lower bound on the asymptotic growth rate of the market share ratio r_t^1/r_t^2. It will be shown that for any investment strategy $(\lambda_t^2)_{t\geq 0}$ the asymptotic growth rate $\lim_{t\to\infty} 1/t \ln(r_t^1/r_t^2) > 0$. This implies (as shown below) that almost surely $r_t^1 \to 1$ and $r_t^2 = 1 - r_t^1 \to 0$. Thus investor 1 (who invests only in the risky asset) gathers total market wealth in the long run.

Consider the right-hand side of (13). Let us first show that for each fixed $d \geq 0$

$$\frac{d + R\,(1-\lambda_{t-1}^2)\,\lambda_t^2\,r_{t-1}^2}{d + R\,(1-\lambda_{t-1}^2)\,r_{t-1}^2} \geq \lambda_t^2 + \alpha \tag{14}$$

with $\alpha \geq 0$ (and $\alpha > 0$ if $d > 0$) for all $\delta \leq \lambda_{t-1}^2 \leq 1-\delta$ and $0 < r_{t-1}^2 \leq 1$. Equation (14) is equivalent to

$$\alpha \leq \frac{(1-\lambda_t^2)\,d}{d + R\,(1-\lambda_{t-1}^2)\,r_{t-1}^2} \tag{15}$$

The right-hand side of (15) is decreasing in r_{t-1}^2 as well as in λ_t^2 and increasing in λ_{t-1}^2. Inserting the maximal resp. minimal possible values for these variables a sufficient condition on α is obtained:

$$\alpha \leq \frac{\delta\, d}{d + R\,(1-\delta)} \tag{16}$$

For each $d \geq 0$, let us define $\alpha(d) \geq 0$ by the right-hand side of (16).

Taking the derivative with respect to d it is straightforward to see that $\alpha(d)$ is increasing in d.

Fixing any $\varepsilon > 0$, (14) thus implies that for every $d(s_t) \geq \varepsilon$,

$$\frac{1}{\lambda_{t-1}^2}\,\frac{d(s_t) + R\,(1-\lambda_{t-1}^2)\,\lambda_t^2}{d(s_t)\, + R\,(1-\lambda_{t-1}^2)} \geq \frac{\lambda_t^2 + \overline{\alpha}}{\lambda_{t-1}^2}$$

with $\overline{\alpha} = \alpha(\varepsilon)$. From (14) and the fact that $\alpha(d) \geq 0$ we also find that for all $d(s_t)$

$$\frac{1}{\lambda_{t-1}^2}\,\frac{d(s_t) + R\,(1-\lambda_{t-1}^2)\,\lambda_t^2}{d(s_t)\, + R\,(1-\lambda_{t-1}^2)} \geq \frac{\lambda_t^2}{\lambda_{t-1}^2}$$

Summarizing these findings, we obtain the following estimate from below on (13):

$$\frac{r_t^1}{r_t^2} \geq \left(\mathbf{1}_{d_t\geq\varepsilon}\,\frac{\lambda_t^2 + \overline{\alpha}}{\lambda_{t-1}^2} + \mathbf{1}_{d_t<\varepsilon}\,\frac{\lambda_t^2}{\lambda_{t-1}^2}\right)\,\frac{r_{t-1}^1}{r_{t-1}^2}$$

where $\mathbf{1}_{d_t\geq\varepsilon} \in \{0,1\}$ with $\mathbf{1}_{d_t\geq\varepsilon} = 1$ if and only if $d(s_t) \geq \varepsilon$. Analogously for $\mathbf{1}_{d_t<\varepsilon}$. Taking logarithms, we find

$$\ln\frac{r_t^1}{r_t^2} \geq \sum_{\tau=1}^{t} \ln\left(\frac{\mathbf{1}_{d_\tau\geq\varepsilon}\,(\lambda_\tau^2 + \overline{\alpha}) + \mathbf{1}_{d_\tau<\varepsilon}\,\lambda_\tau^2}{\lambda_{\tau-1}^2}\right) + \ln\frac{r_0^1}{r_0^2}$$

The sum on the right-hand side can be estimated from below as

$$\begin{aligned}
&\sum_{\tau=1}^{t} \ln\left(\frac{\mathbf{1}_{d_\tau\geq\varepsilon}\,(\lambda_\tau^2 + \overline{\alpha}) + \mathbf{1}_{d_\tau<\varepsilon}\,\lambda_\tau^2}{\lambda_{\tau-1}^2}\right) = \ln\prod_{\tau=1}^{t}\left(\frac{\mathbf{1}_{d_\tau\geq\varepsilon}\,(\lambda_\tau^2 + \overline{\alpha}) + \mathbf{1}_{d_\tau<\varepsilon}\,\lambda_\tau^2}{\lambda_{\tau-1}^2}\right)\\
&\quad= \;\ln\left[\frac{\mathbf{1}_{d_t\geq\varepsilon}\,(\lambda_t^2 + \overline{\alpha}) + \mathbf{1}_{d_t<\varepsilon}\,\lambda_t^2}{\lambda_0^2}\cdot\prod_{\tau=1}^{t-1}\left(\mathbf{1}_{d_\tau\geq\varepsilon}\,\left[1 + \frac{\overline{\alpha}}{\lambda_{\tau-1}^2}\right] + \mathbf{1}_{d_\tau<\varepsilon}\right)\right]\\
&\quad= \;\ln\frac{\mathbf{1}_{d_t\geq\varepsilon}\,(\lambda_t^2 + \overline{\alpha}) + \mathbf{1}_{d_t<\varepsilon}\,\lambda_t^2}{\lambda_0^2} + \sum_{\tau=1}^{t-1}\ln\left(\mathbf{1}_{d_\tau\geq\varepsilon}\,\left[1 + \frac{\overline{\alpha}}{\lambda_{\tau-1}^2}\right] + \mathbf{1}_{d_\tau<\varepsilon}\right)\\
&\quad\geq \;\ln\frac{\lambda_t^2}{\lambda_0^2} + \sum_{\tau=1}^{t-1}\ln\left(\mathbf{1}_{d_\tau\geq\varepsilon}\,\left[1 + \frac{\overline{\alpha}}{1-\delta}\right] + \mathbf{1}_{d_\tau<\varepsilon}\right) \geq \ln\frac{\delta}{1-\delta} + C\,\sum_{\tau=1}^{t-1}\mathbf{1}_{d_\tau\geq\varepsilon}
\end{aligned}$$

where $C = \ln[1 + \overline{\alpha}/(1 - \delta)] > 0$.

The long-run growth rate of r_t^1/r_t^2 is thus bounded from below by

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{r_t^1}{r_t^2} \geq \lim_{t\to\infty} \frac{1}{t} \left(\ln \frac{\delta}{1-\delta} + C \sum_{\tau=1}^{t-1} \mathbf{1}_{d_\tau \geq \varepsilon} + \ln \frac{r_0^1}{r_0^2} \right) = C\, \mathbf{P}\{d(s) \geq \varepsilon\}$$

where the last equality follows from the ergodic theorem.

Assumption (A) implies $\mathbf{P}\{d(s) \geq \varepsilon\} > 0$ for all sufficiently small $\varepsilon > 0$. Since $C > 0$ for every fixed $\varepsilon > 0$, the last equation implies

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{r_t^1}{r_t^2} =: \gamma > 0 \tag{17}$$

for all small enough $\varepsilon > 0$. This means for t large

$$\frac{r_t^1}{1 - r_t^1} = \frac{r_t^1}{r_t^2} \approx \exp(t\,\gamma) \to \infty \text{ as } t \to \infty$$

which implies $r_t^1 \to 1$ (and $r_t^2 \to 0$) as $t \to \infty$ almost surely. Convergence is even exponentially fast. This completes the proof of Theorem 1. □

Remark 1 *We conclude this section with a brief comment on the case of consumption proportional to wealth. Let consumption be given by $c_t^i = (1-\beta) w_t^i$, as e.g. in LeBaron (2001). In this case (1) becomes*

$$w_t^i = \beta \left[R\, m_{t-1}^i + (D_t(s_t) + q_t)\, a_{t-1}^i \right] \tag{18}$$

Repeating essentially the same steps as in the model without consumption, one finds

$$\frac{r_t^1}{r_t^2} = \frac{d(s_t) + \beta\, R\, (1 - \lambda_{t-1}^2)\, \lambda_t^2\, r_{t-1}^2}{d_t(s_t)\, \lambda_{t-1}^2 + R\, (1 - \lambda_{t-1}^2)\, [(1-\beta)\, r_{t-1}^1 + \lambda_{t-1}^2\, r_{t-1}^2]} \cdot \frac{r_{t-1}^1}{r_{t-1}^2} \tag{19}$$

which is identical to (13) for $\beta = 1$, i.e. for $c_t^i = 0$ as considered above.

Now suppose agent 2 becomes extinct, i.e. assume $r_{t-1}^2 \to 0$. Then the first term on the right-hand side of (19) tends to

$$g_t := \frac{d(s_t)}{d_t(s_t)\, \lambda_{t-1}^2 + R\, (1 - \lambda_{t-1}^2)\, (1 - \beta)} \tag{20}$$

If $R(1-\beta) > d_t(s_t)$, then $g_t < 1$. (Note that consumption is required, i.e. $\beta < 1$.) This implies that r_t^1/r_t^2 decreases, i.e. investor 2 gains. Now suppose $R(1-\beta) > d_t(s_t)$ for sufficiently many time periods t to make $\limsup 1/T \ln \sum_{t=0}^{T} g_t < 0$.

Then $r_t^1/r_t^2 \to 0$, i.e. investor 2's market share tends to one. This is a contradiction.

In summary, there are circumstances in which investor 2, who also holds the risk-free asset, is not necessarily wiped out by market forces. Whether the condition on $R(1-\beta) > d_t(s_t)$ is satisfied often enough (along a time path) is an empirical question. The answer may well be a positive one, but it hinges primarily on the interpretation of our highly stylized model.

4 Conclusions

We have shown that while Tobin (1958)'s argument for a liquidity preference as behavior towards risk certainly makes sense in the short run, it is not sustainable in the long run if one takes the wealth dynamics as the criterion. This result has nothing to do with the nature of agents' expectations or the inter-temporal equilibrium concept one evokes. Moreover, our result points to a modeling issue in evolutionary finance models. In these models—depending on the way consumption is modeled—market selection against all investors using the risk-free numeraire asset as a means to transfer wealth over time will prevail. A more robust setting to study the market selection hypothesis is provided in models based on the ideas of Lucas (1978) in which there is a clear distinction between assets to transfer wealth over time and the risk-free numeraire asset that is only used for consumption.

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Appendix (not to be published)

The following elementary calculations show that (10) and (11) are equivalent.

Let $I = 2$. Further, let $\lambda_t^1 = 1$ and $0 < \lambda_t^2 < 1$.
For the first investor $i = 1$ we have

$$w_t^1 = \left(\frac{D_t(s^t)}{\lambda_{t-1}\, w_{t-1}} + \frac{\frac{D_t(s^t)}{\lambda_{t-1}\, w_{t-1}}\, w_{t-1}^1 + \left[R\,(1-\lambda_{t-1}^2) + D_t(s^t)\, \frac{\lambda_{t-1}^2}{\lambda_{t-1}\, w_{t-1}} \right] \lambda_t^2 w_{t-1}^2}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2} \right) w_{t-1}^1$$

A sequence of equivalent expressions is derived by elementary calculations

$$w_t^1 = \left(D_t(s^t) + \frac{D_t(s^t)\, w_{t-1}^1 + \left[R\,(1-\lambda_{t-1}^2)\, \lambda_{t-1}\, w_{t-1} + D_t(s^t)\, \lambda_{t-1}^2 \right] \lambda_t^2 w_{t-1}^2}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2} \right) \frac{w_{t-1}^1}{\lambda_{t-1}\, w_{t-1}}$$

$$\begin{aligned} w_t^1 \;\; = \;\; & \frac{1}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\, \frac{w_{t-1}^1}{\lambda_{t-1}\, w_{t-1}} \times \\ & \Big(D_t(s^t)(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2 \\ & \quad + D_t(s^t)\, w_{t-1}^1 + \left[R\,(1-\lambda_{t-1}^2)\, \lambda_{t-1}\, w_{t-1} + D_t(s^t)\, \lambda_{t-1}^2 \right] \lambda_t^2 w_{t-1}^2 \Big) \end{aligned}$$

$$\begin{aligned} w_t^1 \;\; = \;\; & \frac{1}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\, \frac{w_{t-1}^1}{\lambda_{t-1}\, w_{t-1}} \times \\ & \Big(D_t(s^t)\lambda_{t-1}^2 w_{t-1}^2 - D_t(s^t)\lambda_t^2 \lambda_{t-1}^2 w_{t-1}^2 \\ & \quad + D_t(s^t)\, w_{t-1}^1 + R\,(1-\lambda_{t-1}^2)\, \lambda_{t-1}\, w_{t-1}\, \lambda_t^2 w_{t-1}^2 + D_t(s^t)\, \lambda_{t-1}^2\, \lambda_t^2 w_{t-1}^2 \Big) \end{aligned}$$

$$w_t^1 \;\; = \;\; \frac{1}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\, \frac{w_{t-1}^1}{\lambda_{t-1}\, w_{t-1}} \Big(D_t(s^t)\lambda_{t-1} w_{t-1} + R\,(1-\lambda_{t-1}^2)\, \lambda_{t-1}\, w_{t-1}\, \lambda_t^2 w_{t-1}^2 \Big)$$

Finally, this is equivalent to

$$w_t^1 \;\; = \;\; \frac{D_t(s^t) + R\,(1-\lambda_{t-1}^2)\, \lambda_t^2 w_{t-1}^2}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\, w_{t-1}^1$$

Next, consider the second investor $i = 2$

$$w_t^2 = \Bigg(R\,(1-\lambda_{t-1}^2) + \frac{D_t(s^t)\,\lambda_{t-1}^2}{\lambda_{t-1}\,w_{t-1}} + \lambda_{t-1}^2\,\frac{\frac{D_t(s^t)}{\lambda_{t-1}\,w_{t-1}}\,w_{t-1}^1 + \left[R\,(1-\lambda_{t-1}^2) + D_t(s^t)\,\frac{\lambda_{t-1}^2}{\lambda_{t-1}\,w_{t-1}}\right]\lambda_t^2 w_{t-1}^2}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\Bigg)\,w_{t-1}^2$$

Again we make some elementary transformations

$$w_t^2 = \Bigg(R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}\,w_{t-1} + D_t(s^t)\,\lambda_{t-1}^2 + \lambda_{t-1}^2\,\frac{D_t(s^t)\,w_{t-1}^1 + \left[R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}\,w_{t-1} + D_t(s^t)\,\lambda_{t-1}^2\right]\lambda_t^2 w_{t-1}^2}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\Bigg)\,\frac{w_{t-1}^2}{\lambda_{t-1}\,w_{t-1}}$$

$$w_t^2 = \frac{1}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\,\frac{w_{t-1}^2}{\lambda_{t-1}\,w_{t-1}} \times \Bigg(\left[R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}\,w_{t-1} + D_t(s^t)\,\lambda_{t-1}^2\right](1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2 + \lambda_{t-1}^2\,D_t(s^t)\,w_{t-1}^1 + \lambda_{t-1}^2\left[R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}\,w_{t-1} + D_t(s^t)\,\lambda_{t-1}^2\right]\lambda_t^2 w_{t-1}^2\Bigg)$$

$$w_t^2 = \frac{1}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\,\frac{w_{t-1}^2}{\lambda_{t-1}\,w_{t-1}} \times \Bigg(R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}\,w_{t-1}\,(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2 + D_t(s^t)\,\lambda_{t-1}^2\,(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2 + \lambda_{t-1}^2\,D_t(s^t)\,w_{t-1}^1 + \lambda_{t-1}^2\,R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}\,w_{t-1}\,\lambda_t^2 w_{t-1}^2 + \lambda_{t-1}^2\,D_t(s^t)\,\lambda_{t-1}^2\,\lambda_t^2 w_{t-1}^2\Bigg)$$

$$w_t^2 = \frac{1}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\,\frac{w_{t-1}^2}{\lambda_{t-1}\,w_{t-1}} \times \Bigg(R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}\,w_{t-1}\,\lambda_{t-1}^2 w_{t-1}^2 + D_t(s^t)\,\lambda_{t-1}^2\,\lambda_{t-1}^2 w_{t-1}^2 + \lambda_{t-1}^2\,D_t(s^t)\,w_{t-1}^1\Bigg)$$

$$w_t^2 = \frac{1}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\,\frac{w_{t-1}^2}{\lambda_{t-1}\,w_{t-1}} \times \Bigg(R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}\,w_{t-1}\,\lambda_{t-1}^2 w_{t-1}^2 + D_t(s^t)\,\lambda_{t-1}^2\,\lambda_{t-1} w_{t-1}\Bigg)$$

$$w_t^2 = \frac{1}{(1-\lambda_t^2)\lambda_{t-1}^2 w_{t-1}^2}\, w_{t-1}^2 \left(R\,(1-\lambda_{t-1}^2)\,\lambda_{t-1}^2 w_{t-1}^2 + D_t(s^t)\,\lambda_{t-1}^2 \right)$$

Finally, this is equivalent to

$$w_t^2 = \frac{D_t(s^t) + R\,(1-\lambda_{t-1}^2)\,w_{t-1}^2}{(1-\lambda_t^2)\,w_{t-1}^2}\, w_{t-1}^2$$